

# A general formula for determinants and inverses of $r$ -circulant matrices with third order recurrences

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## Abstract

This note provides formula for determinant and inverse of  $r$ -circulant matrices with general sequences of third order. In other words, the study combines many papers in the literature.

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## 1 Introduction

A  $r$ -circulant matrix of order  $n$ ,  $C_n := \text{circ}_r(c_0, c_1, \dots, c_{n-1})$ , associated with the numbers  $c_0, c_1, \dots, c_{n-1}$ , is defined as

$$C_n = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & \dots & c_0 & c_1 \\ rc_1 & rc_2 & \dots & rc_{n-1} & c_0 \end{pmatrix}. \quad (1)$$

where each row is a cyclic shift of the row above it [1]. If  $r = 1$ , then the matrix  $C_n$  is ordinary circulant matrix. If  $r = -1$ , then the matrix  $C_n$  is skew-circulant matrix.

Circulant matrices and their applications are a fundamental key in many areas of pure and applied science (see [6, 10], and references there in). Recently, many researcher get very interesting properties of them. For example, in [1], Shen and Cen obtained upper and lower bounds for the spectral norms of  $r$ -circulant matrices involving Fibonacci and Lucas numbers. Further, they gave some bounds for the spectral norms of Kronecker and Hadamard products of

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these matrices. In [2], Shen et al. obtained useful formulas for determinants and inverses of circulant matrices with Fibonacci and Lucas numbers, using properties of circulant matrices and this sequences. In [3], Bozkurt and Tam gave formulas for determinants and inverses of circulant matrices involving Jacobsthal and Jacobsthal-Lucas numbers taking into account the method in [2]. Bozkurt and Tam [4] defined  $r$ -circulant matrices with general second order number sequences. Then, the authors obtained formulas for determinant and inverse of this matrix. Moreover, they gave some bounds for norms of  $r$ -circulant matrices involving Fibonacci and Lucas numbers. Yazlık and Taskara [5] considered circulant matrices with  $k$ -Horadam numbers. Then, the authors obtained formulas for determinant and inverse of this matrix. Liu and Jiang [8] defined Tribonacci circulant matrix, Tribonacci left circulant matrix, Tribonacci  $g$ -circulant matrix. Then, the authors acquired determinants and inverses of these matrices. In [9], Bozkurt et al. considered the determinant of circulant and skew-circulant matrices whose entries are Tribonacci numbers. Bozkurt and Yılmaz [11] obtained formulas for determinant and inverse of circulant matrices with Pell and Pell-Lucas numbers.

In this paper, we consider third order linear recurrence for  $n > 2$ :

$$W_n = pW_{n-1} + qW_{n-2} + tW_{n-3} \quad (2)$$

with initial conditions  $W_0 = 0, W_1 = a$  and  $W_2 = b$ . The first few values are

$$0, a, b, pb + qa, p^2b + pqa + qb + ta, \dots$$

Then, we obtain formulas for determinants and inverses of  $r$ -circulant matrices  $E_n$ , i.e.,

$$E_n := \text{circ}_r(W_1, W_2, \dots, W_n),$$

where  $W_n$  is given by (2).

As it can be seen from the definition of the sequence, it is a general form of some well-known sequences. In other words,

◇ If  $p = q = a = b = r = 1$  and  $t = 0$ , then we obtain determinant and inverse of circulant matrices with Fibonacci numbers, as in [2].

◇ If  $p = a = b = r = 1, t = 0$  and  $q = 2$ , then we obtain determinant and inverse of circulant matrices with Jacobsthal numbers, as in [3].

◇ If  $q = a = r = 1, p = b = 2$  and  $t = 0$ , then we obtain determinant and inverse of circulant matrices with Pell numbers, as in [11].

◇ If  $p = q = a = b = t = r = 1$ , then we obtain determinant and inverse of circulant matrices with Tribonacci numbers, as in [8].

◇ If  $p = q = a = b = t = 1$  and  $r = -1$ , then we obtain determinant and inverse of skew-circulant matrices with Tribonacci numbers, as in [9].

To sum up, the derived formulas combine many of the papers in the literature.

## 2 Determinant of $E_n$

This section is dedicated for determinant formula of  $r$ -circulant matrices with general third order sequences. Firstly, let us give the following lemmas.

**Lemma 1** [9] *If*

$$D_n = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ a & b & & & & \\ c & a & b & & & \\ & c & a & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \end{pmatrix}, \quad (3)$$

then

$$\det D_n = \sum_{k=1}^n d_k b^{n-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1} \left(\frac{a}{2\sqrt{bc}}\right), \quad (4)$$

where  $U_k(x)$  is the  $k$ th Chebyshev polynomial of second kind.

**Lemma 2** *If*

$$B_n = \begin{pmatrix} X_1 & d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ Y_1 & f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ 0 & a & b & 0 & & & 0 \\ & c & a & b & & & \\ & & c & a & \ddots & & \\ & & & \ddots & \ddots & \ddots & 0 \\ 0 & & & 0 & c & a & b \end{pmatrix},$$

then

$$\det(B_n) = X_1 \sum_{k=1}^{n-1} f_k b^{n-1-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1} \left(\frac{a}{2\sqrt{bc}}\right) - Y_1 \sum_{k=1}^{n-1} d_k b^{n-1-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1} \left(\frac{a}{2\sqrt{bc}}\right),$$

where  $U_k(x)$  is the  $k$ th Chebyshev polynomial of second kind.

**Proof.** Using the same method in the first Lemma 1, we have

$$\det(B_n) = X_1 \det(F_{n-1}) - Y_1 \det(D_{n-1}) .$$

So, this proof is completed. ■

**Theorem 3** *For  $n \geq 4$ , the determinant of  $E_n$  is*

$$\begin{aligned} & W_1 \left[ (g_n + j f_n) \left( (W_1 - r(pW_n + qW_{n-1}))x_n^{n-3} + rt \sum_{k=2}^{n-2} W_{n-1-k} x_n^{n-2-k} (-\sqrt{x_n z_n})^{k-1} U_{k-1} \left( \frac{y_n}{2\sqrt{x_n z_n}} \right) \right) \right. \\ & \left. - h_n \sum_{k=1}^{n-2} [rW_{n+1-k} - (pr - j)W_{n-k}] x_n^{n-2-k} (-\sqrt{x_n z_n})^{k-1} U_{k-1} \left( \frac{y_n}{2\sqrt{x_n z_n}} \right) \right], \end{aligned}$$

where  $x_n = W_1 - rW_{n+1}$ ,  $y_n = W_2 - rW_{n+2} - p(W_1 - rW_{n+1})$ ,  $z_n = -rtW_n$ ,  
 $j = -\frac{r(W_2 - pW_1)}{W_1}$  and

$$\begin{aligned} f_n &= \sum_{i=2}^n W_i e^{n-i}, \\ g_n &= r \sum_{i=2}^{n-1} (W_{i+1} - pW_i) e^{n-i} + W_1 - prW_n, \\ h_n &= rt \sum_{i=1}^{n-3} W_i e^{n-1-i} + (W_1 - r(pW_n + qW_{n-1})e + W_2 - pW_1 - qrW_n). \end{aligned}$$

**Proof.** Firstly, let us define  $n$ -square matrix

$$F_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{n-2} & 0 & \dots & 0 & 1 \\ 0 & e^{n-3} & 0 & \dots & 1 & 0 \\ 0 & e^{n-4} & 0 & & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & e & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (5)$$

here  $e$  is the positive root of the characteristic equation  $x_n e^2 + y_n e + z_n = 0$ ,  
i.e.,

$$e = \frac{-y_n + \sqrt{y_n^2 - 4x_n z_n}}{2x_n},$$

where

$$x_n = W_1 - rW_{n+1}, \quad y_n = W_2 - rW_{n+2} - p(W_1 - rW_{n+1}), \quad \text{and } z_n = -rtW_n.$$

Then, consider  $n$ -square matrix  $G_n$  as below:

$$G_n = \left( \begin{array}{c|cccccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -pr & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ -qr & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -p \\ -tr & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -p & -q \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -p & -q & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -p & -q & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \ddots & -q & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -p & \ddots & -t & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -p & -q & -t & \dots & 0 & \vdots & \vdots & \vdots \\ 0 & 1 & -p & -q & -t & 0 & \dots & 0 & 0 & 0 & 0 \end{array} \right)$$

It can be seen that for all  $n > 3$ ,

$$\det(G_n) = \det(F_n) = \begin{cases} 1, & n \equiv 1, 2 \pmod{4} \\ -1, & n \equiv 1, 2 \pmod{4}, \end{cases}$$

where  $F_n$  is defined in (5) and  $\det(G_n F_n) = 1$ . By matrix multiplication, we get;

$$K_n = G_n E_n F_n, \quad (6)$$

i.e.,

$$K_n = \left( \begin{array}{cc|cccccc} W_1 & f_n & W_{n-1} & W_{n-2} & \cdots & W_2 \\ r(W_2 - pW_1) & g_n & r(W_n - pW_{n-1}) & r(W_{n-1} - pW_{n-2}) & \cdots & r(W_3 - pW_2) \\ 0 & h_n & W_1 - r(pW_n + qW_{n-1}) & rtW_{n-3} & \cdots & rtW_1 \\ \hline 0 & 0 & y_n & x_n & & \\ 0 & 0 & z_n & y_n & x_n & \\ & & & \ddots & \ddots & \ddots \\ & & & & z_n & y_n & x_n \end{array} \right),$$

where

$$f_n = \sum_{i=2}^n W_i e^{n-i},$$

$$g_n = W_1 - prW_n + r \sum_{i=2}^{n-1} (W_{i+1} - pW_i) e^{n-i},$$

$$h_n = W_2 - pW_1 - qrW_n + (W_1 - r(pW_n + qW_{n-1}))e + rt \sum_{i=1}^{n-3} W_i e^{n-1-i}.$$

Multiplying the first row with  $j = -\frac{r(W_2 - pW_1)}{W_1}$  and adding it to the second

row in  $K_n$ , we obtain

$$|K_n| = \left| \begin{array}{cc|cccccc} W_1 & f_n & W_{n-1} & W_{n-2} & \cdots & W_2 \\ 0 & g_n + jf_n & rW_n + (j - rp)W_{n-1} & rW_{n-1} + (j - rp)W_{n-2} & \cdots & rW_3 + (j - rp)W_2 \\ 0 & h_n & W_1 - r(pW_n + qW_{n-1}) & rtW_{n-3} & \cdots & rtW_1 \\ \hline 0 & 0 & y_n & x_n & \cdots & 0 \\ \vdots & \vdots & z_n & y_n & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & z_n & y_n & x_n \end{array} \right|.$$

By Laplace expansion on the first column

$$\det K_n = W_1 \det Z_n = \det E_n$$

here

$$Z_n = \left( \begin{array}{cc|cccccc} g_n + jf_n & rW_n + (j - rp)W_{n-1} & rW_{n-1} + (j - rp)W_{n-2} & \cdots & rW_3 + (j - rp)W_2 \\ h_n & W_1 - r(pW_n + qW_{n-1}) & rtW_{n-3} & \cdots & rtW_1 \\ \hline 0 & y_n & x_n & 0 & 0 \\ 0 & z_n & y_n & x_n & & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & z_n & y_n & x_n \end{array} \right).$$

Applying Lemma 2, we complete the proof. ■

### 3 Inverse of $E_n$

In this section, we compute the inverse of the matrix  $E_n$ . Note that, just only for the inverse, we consider  $W_2 = pa$ . So,

$$G_n E_n F_n = K_n = \left( \begin{array}{cc|cccccc} W_1 & f_n & W_{n-1} & W_{n-2} & \cdots & W_2 \\ 0 & g_n & r(W_n - pW_{n-1}) & r(W_{n-1} - pW_{n-2}) & \cdots & r(W_3 - pW_2) \\ 0 & h_n & W_1 - r(pW_n + qW_{n-1}) & rtW_{n-3} & \cdots & rtW_1 \\ \hline 0 & 0 & y_n & x_n & & \\ 0 & 0 & z_n & y_n & x_n & \\ & & & \ddots & \ddots & \ddots \\ 0 & 0 & & & z_n & y_n & x_n \end{array} \right).$$

**Lemma 4** [8] Let  $\psi = \begin{pmatrix} \alpha & V \\ U & A \end{pmatrix}$  be an  $(n-2)$ -square matrix, then

$$\psi^{-1} = \begin{pmatrix} \frac{1}{l} & -\frac{1}{l}VA^{-1} \\ -\frac{1}{l}A^{-1}U & A^{-1} + \frac{1}{l}A^{-1}UVA^{-1} \end{pmatrix},$$

where  $l = \alpha - VA^{-1}U$ ,  $V$  is a row vector and  $U$  is a column vector.

**Lemma 5** Let us define the matrix  $T = [t_{i,j}]_{i,j=1}^{n-3}$  of the form:

$$t_{ij} = \begin{cases} W_1 x_n & , i = j, \\ W_1 y_n & , i = j + 1, \\ W_1 z_n & , i = j + 2, \\ 0 & , \text{otherwise.} \end{cases}$$

Then, inverse of  $T$  is

$$T^{-1} = [t'_{i,j}]_{i,j=1}^{n-3} = \begin{cases} \frac{1}{W_1 x_n} & , i = j \\ -\frac{y_n}{W_1 x_n^2} & , i = j + 1 \\ -\frac{y_n t'_{i-2,j} + z_n t'_{i-1,j}}{x_n} & , i = j + k (k \geq 2) \\ 0 & , i < j. \end{cases} \quad (7)$$

**Proof.** From matrix multiplication, we can easily see that  $TT^{-1} = T^{-1}T = I_{n-3}$ , where  $I_{n-3}$  is identity matrix. ■

**Theorem 6** Let  $E_n = \text{circ}_r(W_1, W_2, \dots, W_n)$  be  $r$ -circulant matrix. Then,

$$E_n^{-1} = \text{circ}_r \left( c'_2 - \left( p + \frac{h_n}{g_n} \right) c'_3 - qc'_4 - tc'_5, \right. \\ \left. -pc'_2 + \left( \frac{ph_n}{g_n} - q \right) c'_3 - tc'_4, \frac{c'_n}{r}, \frac{c'_{n-1} - pc'_n}{r} \right. \\ \left. \frac{1}{r} (c'_{n-2} - pc'_{n-1} - qc'_n), \dots, \frac{1}{r} (c'_{n-k+3} - pc'_{n-k+4} - qc'_{n-k+5} - tc'_{n-k+6}) \right),$$

where

$$\begin{aligned}
c'_1 &= 0, \\
c'_2 &= W_1^2 g_n, \\
c'_3 &= -\frac{rW_1}{g_n} \sum_{k=0}^{n-3} s_k (W_{n-k} - pW_{n-k-1}), \quad (\text{for } s_0 = \frac{1}{t}), \\
c'_4 &= -\frac{rp_1 W_1 (W_n - pW_{n-1})}{g_n} - \frac{rW_1}{g_n} \sum_{k=1}^{n-3} u_{k,1} (W_{n-k} - pW_{n-k-1}), \\
&\vdots \\
c'_t &= -\frac{rp_{t-3} W_1 (W_n - pW_{n-1})}{g_n} - \frac{rW_1}{g_n} \sum_{k=1}^{n-3} u_{k,t-3} (W_{n-k} - pW_{n-k-1}), \quad (t \geq 4)
\end{aligned}$$

and

$$\begin{aligned}
g_n &= W_1 - prW_n + r \sum_{i=2}^{n-1} (W_{i+1} - pW_i) e^{n-i}, \\
h_n &= W_2 - pW_1 - qrW_n + (W_1 - r(pW_n + qW_{n-1}))e + rt \sum_{i=1}^{n-3} W_i e^{n-1-i}.
\end{aligned}$$

**Proof.** Firstly, Let us define

$$H_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{h_n}{g_n} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$L_n = \begin{pmatrix} W_1 & -f_n & -W_{n-1} + \frac{rf_n(W_n - pW_{n-1})}{g_n} & -W_{n-2} + \frac{rf_n(W_{n-1} - pW_{n-2})}{g_n} & \cdots & -W_{n-2} + \frac{rf_n(W_3 - pW_2)}{g_n} \\ 0 & W_1 & -\frac{rW_1(W_n - pW_{n-1})}{g_n} & -\frac{rW_1(W_{n-1} - pW_{n-2})}{g_n} & \cdots & -\frac{rW_1(W_3 - pW_2)}{g_n} \\ 0 & 0 & W_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & W_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & W_1 \end{pmatrix}.$$

Then, from matrix multiplication, we have

$$H_n G_n E_n F_n L_n = \begin{pmatrix} W_1^2 & 0 & & & & \\ 0 & W_1 g_n & & & & \\ & & W_1 \rho_3 & W_1 \rho_4 & \cdots & W_1 \rho_n \\ & & W_1 y_n & W_1 x_n & & \\ & & W_1 z_n & W_1 y_n & \ddots & \\ & & & \ddots & \ddots & W_1 x_n \\ & & & & W_1 z_n & W_1 y_n \end{pmatrix} = \mathcal{Y}_1 \oplus N,$$

where  $\mathcal{Y}_1 = \text{diag}(W_1^2, W_1 g_n)$ ,  $\mathcal{Y}_1 \oplus N$  is the direct sum of  $\mathcal{Y}_1$  and  $N$ ,

$$\rho_3 = W_1 - r \left( W_n \left( p + \frac{h_n}{g_n} \right) - W_{n-1} \left( q + p \frac{h_n}{g_n} \right) \right)$$

and

$$\rho_i = -\frac{r h_n}{g_n} (W_{n-i+3} - p W_{n-i+2}) + r t W_{n-i+1} \quad \text{for } i = 4, 5, \dots, n.$$

If we define  $P = H_n G_n$  and  $Q = F_n L_n$ , we get

$$E_n^{-1} = Q (\mathcal{Y}_1^{-1} \oplus N^{-1}) P.$$

According to Lemma 4, we define  $(n-2)$ -square matrix

$$N = \begin{pmatrix} W_1 \rho_3 & V \\ U & T \end{pmatrix}.$$

Then, we have

$$N^{-1} = \begin{pmatrix} \frac{1}{l} & \frac{-VT^{-1}}{l} \\ \frac{-T^{-1}U}{l} & T^{-1} + \frac{1}{l} T^{-1} U V T^{-1} \end{pmatrix},$$

where

$$\begin{aligned} U &= (W_1 y_n, W_1 z_n, 0, \dots, 0)^T, \\ V &= (W_1 \rho_4, W_1 \rho_5, \dots, W_1 \rho_n), \\ T &= \begin{cases} W_1 x_n & , i = j \\ W_1 y_n & , i = j + 1 \\ W_1 z_n & , i = j + 2 \\ 0 & , \text{otherwise}, \end{cases} \\ l &= W_1 \left( \rho_3 - W_1 y_n \sum_{i=1}^{n-3} t_{i1} \rho_{i+3} - W_1 z_n \sum_{i=1}^{n-4} \rho_{i+4} \right). \end{aligned}$$

Let be  $R = \frac{-VT^{-1}}{l}$  row vector,  $S = \frac{-T^{-1}U}{l}$  column vector and  $J = T^{-1} + \frac{1}{l} T^{-1} U V T^{-1}$ , where  $T^{-1}$  is as in (7) Then, we have

$$R = [p_1, p_2, \dots, p_{n-3}],$$

where  $p_i = \frac{-W_1}{l} \sum_{k=i}^{n-3} \rho_{k+3} t'_{k,i}$ ,

$$S = [s_1, s_2, \dots, s_{n-3}]^T,$$

where  $s_1 = \frac{-W_1}{l} t'_{1,1} y_n$  and for  $i \geq 2$ ,  $s_i = \frac{-W_1}{l} (y_n t'_{i,1} + z_n t'_{i,2})$ ,

$$J = u_{i,j} = \begin{cases} t'_{1,j} - \frac{W_1^2}{l} y_n t'_{1,1} \sum_{k=j}^{n-3} \rho_{k+3} t'_{k,j} & , \text{ for } i = 1 \\ t'_{i,j} - \frac{W_1^2}{l} (y_n t'_{i,1} + z_n t'_{i,2}) \sum_{k=j}^{n-3} \rho_{k+3} t'_{k,j} & , \text{ for } i = 2, 3, \dots, n-3. \end{cases}$$



So, we obtain

$$N^{-1} = \begin{pmatrix} \frac{1}{l} & p_1 & p_2 & \cdots & p_{n-3} \\ s_1 & u_{1,1} & u_{1,2} & \cdots & u_{1,n-3} \\ s_2 & u_{2,1} & u_{2,2} & \cdots & u_{2,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-3} & u_{n-3,1} & u_{n-3,2} & \cdots & u_{n-3,n-3} \end{pmatrix}_{(n-2) \times (n-2)},$$

where  $s_i$ 's,  $p_i$ 's and  $u_{i,j}$ 's are as in above.

The last row elements of the  $Q = F_n L_n$  are 0,  $W_1$ ,  $-\frac{rW_1(W_n - pW_{n-1})}{g_n}$ ,  $-\frac{rW_1(W_{n-1} - pW_{n-2})}{g_n}$ ,  $\dots$ ,  $-\frac{rW_1(W_3 - pW_2)}{g_n}$ . Then, the last row elements of  $Q(\mathcal{Y}_1^{-1} \oplus N^{-1})$  are as the following:

$$\begin{aligned} c'_1 &= 0, \\ c'_2 &= W_1^2 g_n, \\ c'_3 &= -\frac{rW_1}{g_n} \sum_{k=0}^{n-3} s_k (W_{n-k} - pW_{n-k-1}), \quad (\text{for } s_0 = \frac{1}{l}), \\ c'_4 &= -\frac{rp_1 W_1 (W_n - pW_{n-1})}{g_n} - \frac{rW_1}{g_n} \sum_{k=1}^{n-3} u_{k,1} (W_{n-k} - pW_{n-k-1}), \\ &\vdots \\ c'_t &= -\frac{rp_{t-3} W_1 (W_n - pW_{n-1})}{g_n} - \frac{rW_1}{g_n} \sum_{k=1}^{n-3} u_{k,t-3} (W_{n-k} - pW_{n-k-1}), \quad (t \geq 4). \end{aligned}$$

Since inverse of  $r$ -circulant matrix is  $r$ -circulant matrix [4],  $E_n^{-1}$  matrix is an  $r$ -circulant matrix. If  $E_n^{-1} = \text{circ}_r(c_1, c_2, \dots, c_n)$ , last row elements of the  $E_n^{-1}$  matrix are as in below:

$$\begin{aligned} rc_2 &= -prc'_2 + \left(\frac{prh_n}{g_n} - qr\right) c'_3 - trc'_4 \\ rc_3 &= c'_n \\ rc_4 &= c'_{n-1} - pc'_n \\ rc_5 &= c'_{n-2} - pc'_{n-1} - qc'_n \\ &\vdots \\ rc_k &= c'_{n-k+3} - pc'_{n-k+4} - qc'_{n-k+5} - tc'_{n-k+6} \quad (\text{for } 5 < k \leq n), \\ c_1 &= c'_2 - \left(p + \frac{h_n}{g_n}\right) c'_3 - qc'_4 - tc'_5. \end{aligned}$$

Therefore, we complete this proof. ■

## References

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